Note on the minimal size of a graph with generalized connectivity $\kappa_3 = 2$*

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Abstract

The concept of generalized k-connectivity $\kappa_k(G)$ of a graph G was introduced by Chartrand et al. in recent years. In our early paper, extremal theory for this graph parameter was started. We determined the minimal number of edges of a graph of order n with $\kappa_3 = 2$, i.e., for a graph G of order n and size $e(G)$ with $\kappa_3(G) = 2$, we proved that $e(G) \geq \frac{6}{5}n$, and the lower bound is sharp by constructing a class of graphs, only for $n \equiv 0$ $(mod\ 5)$ and $n \neq 10$. In this paper, we improve the lower bound to $\lceil \frac{6}{5}n \rceil$. Moreover, we show that for all $n \geq 4$ but $n = 9, 10$, there always exists a graph of order n with $\kappa_3 = 2$ whose size attains the lower bound $\lceil \frac{6}{5}n \rceil$. Whereas for $n = 9, 10$ we give examples to show that $\lceil \frac{6}{5}n \rceil + 1$ is the best possible lower bound. This gives a clear picture on the minimal size of a graph of order n with generalized connectivity $\kappa_3 = 2$.

Keywords: k-connectivity; internally disjoint trees

AMS Subject Classification 2010: 05C40, 05C05.

1 Introduction

We follow the terminology and notations of [1], and all graphs considered here are always finite and simple. As usual, we denote the numbers of vertices and edges in G

*Supported by NSFC and the Fundamental Research Funds for the Central Universities.

by $n(G)$ and $e(G)$ (or simply n and e), and these two basic parameters are called the *order* and *size* of G, respectively. A stable set in a graph is a set of vertices no two of which are adjacent. A vertex with degree one in a tree is called a leaf. The *connectivity* $\kappa(G)$ of a graph G is defined as the minimum cardinality of a set Q of vertices of G such that $G - Q$ is disconnected or trivial. A well-known theorem of Whitney [5] provides an equivalent definition of the connectivity. For each 2-subset $S = \{u, v\}$ of vertices of G, let $\kappa(S)$ denote the maximum number of internally disjoint uv-paths in G. Then $\kappa(G) = \min\{\kappa(S)\}$, where the minimum is taken over all 2-subsets S of $V(G)$.

In [2], the authors generalized the concept of connectivity as follows. Let G be a nontrivial connected graph of order n and k an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for every pair i, j of distinct integers with $1 \leq i, j \leq \ell$ (note that the trees are vertex-disjoint in $G \backslash S$). The *k-connectivity*, denoted by $\kappa_k(G)$, of G is then defined by $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k-subsets S of $V(G)$. Obviously, $\kappa_2(G) = \kappa(G)$.

This paper is a further development of our early work [3], where we determined the minimal number of edges of a graph with $\kappa_3 = 2$, i.e., for a graph G of order n and size $e(G)$ with $\kappa_3(G) = 2$, we proved that $e(G) \geq \frac{6}{5}n$, and the lower bound is sharp by constructing a class of graphs, only $n \equiv 0$ $(mod\ 5)$ and $n \neq 10$. Note that the number of edges is integral and so the order of the graph attaining the lower bound must be a multiple of 5. On the other hand, since $e(G)$ is an integer, the lower bound can be naturally improved to $\lceil \frac{6}{5}n \rceil$. In this paper, we want to show that for all $n \geq 4$ but $n = 9, 10$, the lower bound $\lceil \frac{6}{5}n \rceil$ is best possible, whereas for $n = 9, 10$ we give examples to show that $\lceil \frac{6}{5}n \rceil + 1$ is the best possible lower bound. This gives a clear picture on the minimal size of a graph of order n with generalized connectivity $\kappa_3 = 2$.

2 Preliminaries

Before proceeding, we list some known results in [3] and [4].

Lemma 2.1 ([4]). *If G is a connected graph with minimum degree δ, then $\kappa_3(G) \leq \delta$. In particular, if there are two adjacent vertices of degree δ, then $\kappa_3(G) \leq \delta - 1$.*

Lemma 2.2 ([3]). *For a positive integer $k \neq 2$, let $C = x_1y_1x_2y_2 \ldots x_{2k}y_{2k}x_1$ be a cycle of length $4k$. Add k new vertices $z_1, z_2, \ldots, z_k$ to C, and join z_i to x_i and x_{i+k}, for $1 \leq i \leq k$. The resulting graph is denoted by H. Then, the 3-connectivity of H is 2,*

namely, $\kappa_3(H) = 2$*.*

Lemma 2.3 ([3])**.** *For any connected graph* G *of order* 10 *and size* 12*,* $\kappa_3(G) = 1$*.*

Remark 2.1: Note that there exists a graph G such that $n = 10$, $e(G) = 13$ and $\kappa_3(G) = 2$, as shown in Figure 1.

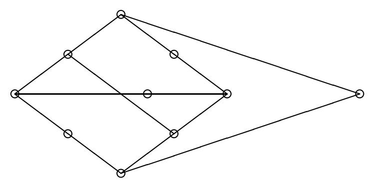

Figure 1: The graph G of order 10 and size 13 with $\kappa_3(G) = 2$.

Now we turn to the graphs of order 9 and size 11.

Lemma 2.4. *For any connected graph* G *of order* 9 *and size* 11*,* $\kappa_3(G) = 1$*.*

Proof. Assume, to the contrary, that there is a connected graph G of order $n = 9$ and size $m = 11$ with $\kappa_3(G) = 2$. By Lemma 2.1, we have the minimum degree $\delta(G) \geq 2$. Denote by X the set of vertices of degree 2 in G. It follows that $2m = \Sigma_{v \in V(G)} d(v) \geq 2|X| + 3(n - |X|)$, namely, $|X| \geq 3n - 2m = 5$. On the other hand, by Lemma 2.1 again, we get that X is a stable set. Let m' be the number of edges joining two vertices belonging to Y, where $Y = V(G) - X$. It is clear that $m = 2|X| + m'$. So $|X| \leq \frac{m}{2} = 5.5$. Now we can conclude that $|X| = 5$, $|Y| = 4$, $m' = 1$ and every vertex in Y has degree exactly 3. Set $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4\}$. Since $m' = 1$, without loss of generality, suppose that $y_1 y_2$ is the only edge.

Case 1: There is a vertex in X that is adjacent to both y_1 and y_2.

Note that G is a simple connected graph and every vertex in X has degree 2. It is not hard to get that G is isomorphic to the graph as shown in Figure 2. Then observe that it is impossible to find two internally-disjoint trees connecting the vertices x_1, x_2 and x_4, contrary to our assumption.

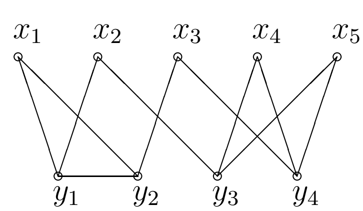

Figure 2: The graph for Case 1 of Lemma 2.4

Case 2: There is no vertex in X that is adjacent to both y_1 and y_2.

Subcase 2.1: For every 2-subset $\{y_i, y_j\}$ of Y other than $\{y_1, y_2\}$, there is a vertex in X that is adjacent to both y_i and y_j, where $1 \leq i \neq j \leq 5$.

Note that there are exactly five vertices in X and five 2-subsets of Y other than $\{y_1, y_2\}$, namely, $\{y_1, y_3\}, \{y_1, y_4\}, \{y_2, y_3\}$, $\{y_2, y_4\}, \{y_3, y_4\}$. Thus, we may assume that G is isomorphic to the graph as shown in Figure 3. Consider the three vertices x_1, x_2 and x_5, and we can get $\kappa_3(G) = 1$, contrary to our assumption.

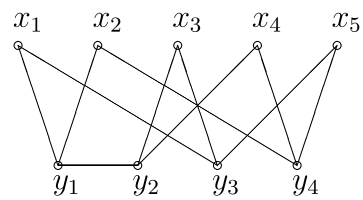

Figure 3: The graph for Subcase 2.1 of Lemma 2.4

Subcase 2.2: Except $\{y_1, y_2\}$, there exists another 2-subset such that no vertex in X is adjacent to both of the vertices in that subset.

In such a situation, there must exist some 2-subset $\{y_i, y_j\}$ such that at least two vertices in X are adjacent to both y_i and y_j, where $1 \leq i \neq j \leq 5$. If $\{y_i, y_j\} = \{y_3, y_4\}$, it is not hard to get that there must exist a vertex in X that is adjacent to both y_1 and y_2, contrary to the case. So without loss of generality, we may assume that $\{y_i, y_j\} = \{y_1, y_3\}$. Then we can get G is isomorphic to the graph as shown in Figure 4. Observe that it is impossible to find two internally-disjoint trees connecting the vertices x_1, x_4 and x_5, contrary to our assumption.

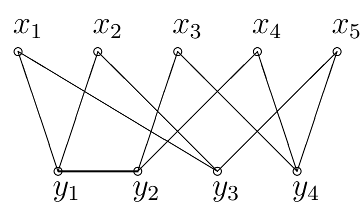

Figure 4: The graph for Subcase 2.2 of Lemma 2.4

The proof is complete. ∎

Remark 2.2: Notice that there exists a graph G such that $n = 9$, $e(G) = 12$ and $\kappa_3(G) = 2$, as shown in Figure 5.

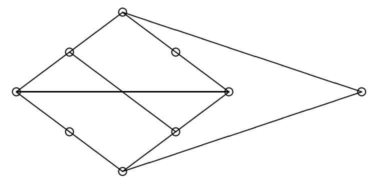

Figure 5: The graph G of order 9 and size 12 with $\kappa_3(G) = 2$.

Next we describe an operation on a vertex of degree 2.

For a vertex u of degree 2, to *smooth* u is to delete u and then add an edge between its neighbors. Obviously, performing such an operation, the numbers of vertices and edges decrease by one, respectively. Moreover, the degrees of the remaining vertices are not changed.

Lemma 2.5. *Let G be a graph such that the set X of vertices of degree 2 is nonempty. Denote by G' the new graph obtained by smoothing a vertex in X, and then we have $\kappa_3(G') \geq \kappa_3(G)$.*

Proof. Let u be a vertex in X and $\{w_1, w_2\}$ the neighbor set of u. Suppose that G' is obtained by smoothing u. Clearly, $V(G') = V(G) - u$. For any three vertices v_1, v_2 and v_3 of G', let $S = \{v_1, v_2, v_3\}$. Obviously, $S \subseteq V(G)$. Let T be a tree connecting S in G. Note that if v is a leaf of T, we can assume that $v \in S$. Otherwise, $T' = T - v$ is still a tree connecting S and uses less vertices. Now if $u \in V(T)$, then we can see that $T' = T - u + w_1w_2$ is exactly a tree connecting S in G'. If $u \notin V(T)$, the operation of smoothing u has nothing to do with T and so T is still a tree connecting S in G'. Therefore, it is not hard to get that $\kappa_{G'}(S) \geq \kappa_G(S)$. From the definition of κ_3, the conclusion that $\kappa_3(G') \geq \kappa_3(G)$ follows. ∎

Remark 2.3: For a given G, if we successively do the operation of smoothing a vertex of degree 2 more than once, the final graph is denoted by G'. We can also get $\kappa_3(G') \geq \kappa_3(G)$.

3 Lower bound

Lemma 3.1 ([3]). *If G is a graph of order n with $\kappa_3(G) = 2$, then $e(G) \geq \frac{6}{5}n$ and the lower bound is sharp.*

Note that the number of edges is integral and so the order of the graph attaining the lower bound must be a multiple of 5. In [3], we showed that for all positive integer k

other than 2, there exists a graph of order $n = 5k$ which attains the lower bound. On the other hand, since $e(G)$ is an integer, the lower bound can be improved to $\lceil \frac{6}{5}n \rceil$. Naturally, we want to know whether there is a graph of order n attaining the lower bound for any positive integer n.

Theorem 3.1. *If G is a graph of order n with $\kappa_3(G) = 2$, then $e(G) \geq \lceil \frac{6}{5}n \rceil$. Moreover, the lower bound is sharp for all $n \geq 4$ and $n \neq 9, 10$.*

Proof. Since the number of edges must be an integer, by Lemma 3.1, the lower bound $\lceil \frac{6}{5}n \rceil$ is obvious.

Note that all graphs considered here are always simple. Therefore, any graph attaining the lower bound must have at least four vertices. Moreover, by Lemmas 2.3 and 2.4, we know that there is no simple connected graph G of order 9 and size 11 or order 10 and size 12 such that $\kappa_3(G) = 2$.

For $n = 8$, there is a graph G' of order n such that $\kappa_3(G') = 2$ as shown in Figure 6. Moreover, $e(G') = 10 = \lceil \frac{6}{5} \times 8 \rceil$, which means that G' attains the lower bound for $n = 8$.

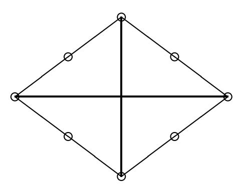

Figure 6: The graph G' attaining the lower bound for $n = 8$

Now, smooth a vertex of degree 2 in G'. Clearly, the resulting graph G'' is simple and $\delta(G'') = 2$. By Lemma 2.5, we can get $\kappa_3(G'') \geq (\kappa_(G') = 2)$ and so clearly $\kappa_3(G'') = 2$. Moreover, $n = 8 - 1 = 7$ and $e = 10 - 1 = 9 = \lceil \frac{6}{5} \times 7 \rceil$. The graph G'' is what we want to find for $n = 7$. Similarly, the graph obtained from G'' by smoothing any one vertex of degree 2 attains the lower bound for $n = 6$.

Next, we consider the graph H in Lemma 2.2. In [3], We obtained that $\kappa_3(H) = 2$, $n(H) = 5k$ and $e(H) = 6k$, for $k \neq 2$. So H is exactly the graph of order $n = 5k$ which attains the lower bound.

For $k \geq 3$, let $k' = k - 1$ and then $n(H) = 5k' + 5$ and $e(H) = 6k' + 6$. Let X be the set of vertices of degree 2. Clearly $|X| = 3k' + 3 > 4$, where $k' \geq 2$. Now for the graph H, smooth successively any t vertices in X, for $1 \leq t \leq 4$. For any t, it is easy to check that no parallel edge can arise. Moreover, since $|X| > 4$, the minimum degree of the resulting graph H' is still 2. Combining Lemma 2.1 and Remark 2.3, we can get the 3-connectivity

of the resulting graph H' is 2. Now let us consider the numbers of vertices and edges of H'.

When $t = 1$, $n(H') = 5k' + 4$ and $e(H') = 6k' + 5 = \lceil \frac{6}{5}(5k' + 4) \rceil$;

When $t = 2$, $n(H') = 5k' + 3$ and $e(H') = 6k' + 4 = \lceil \frac{6}{5}(5k' + 3) \rceil$;

When $t = 3$, $n(H') = 5k' + 2$ and $e(H') = 6k' + 3 = \lceil \frac{6}{5}(5k' + 2) \rceil$;

When $t = 4$, $n(H') = 5k' + 1$ and $e(H') = 6k' + 2 = \lceil \frac{6}{5}(5k' + 1) \rceil$.

Note that $k' \geq 2$. Therefore, for all $n \geq 4$ but $n = 9, 10$, we can always find a graph of order n attaining the lower bound. ∎

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